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A Variational Multigrid Framework for Multiscale Fault Systems with Rate- and State-Dependent Friction: Convergence Analysis and Adaptive Solution Strategies

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Abstract :*We present a comprehensive variational framework for the numerical simulation of multiscale geological fault systems governed by rate- and state-dependent friction laws. Extending the variational approach for subduction zones to layered fault networks comprising multiple non-intersecting interfaces, we derive a mathematical model that encompasses both Dieterich- and Ruina-type friction as special cases while accommodating large tangential displacements. Semi-discretization in time via an energy-conserving Newmark scheme yields a coupled system of non-smooth convex minimization problems at each time step. Spatial discretization employs a dual-mortar finite element method with a piecewise-constant state approximation, enabling efficient decoupling via fixed-point iteration. For the resulting rate problems, we establish global convergence of truncated non-smooth Newton multigrid (TNNMG) methods and prove stability of the fully discrete scheme under standard regularity assumptions. Numerical experiments on spring-slider and layered five-body configurations demonstrate the robustness of the solver with respect to the number of faults; adaptive time-stepping captures slip events across multiple orders of magnitude in velocity variation.*

Keywords: *Rate- and State-Dependent Friction; Multibody Contact; Mortar Methods; Non-Smooth Multigrid; Variational Inequalities; Adaptive Time-Stepping*

INTRODUCTION

Stress accumulation and release in geological fault networks constitute fundamental mechanisms governing earthquake dynamics. The phenomenology spans a broad range of tectonic settings—from subduction zones such as the Nazca plate and strike-slip faults such as the San Andreas fault, to complex multiscale systems exemplified by the Atacama fault zone (Ampuero & Rubin, 2008; Pipping, Sander, & Kornhuber, 2015; Pipping, Kornhuber, Rosenau, & Oncken, 2016). The strongly varying temporal scales between interseismic loading phases and coseismic slip events necessitate complementary numerical approaches alongside experimental, field, and laboratory studies (Barbot, Lapusta, & Avouac, 2012).

The mathematical description of frictional processes along faults was fundamentally advanced by the Dieterich–Ruina model of rate- and state-dependent friction (RSF) (Ruina, 1983), which has since become the standard for characterizing fault slip behavior (Ampuero & Rubin, 2008; Pipping et al., 2015; Pipping et al., 2016). This framework extends classical Tresca friction by introducing a friction coefficient $\mu = \mu(V, \theta)$ that depends on the slip rate V and an internal state variable θ capturing memory effects in the frictional interface. The variational structure of RSF was first identified and exploited by Pipping et al. (2015), opening avenues for robust numerical treatment through convex analysis.

Dynamic rupture simulation has a substantial history in computational seismology (Barbot et al., 2012; De la Puente, Ampuero, & Käser, 2009; Pelties et al., 2012). De la Puente et al. (2009) developed discontinuous Galerkin schemes with arbitrary high-order (ADER) time integration for dynamic slip events, later extended to three dimensions (Pelties et al., 2012) and implemented in the SeisSol software package—successfully applied to the 2016 Kaikōura earthquake cascade (Ulrich et al., 2019). Alternative diffuse representations of faults have been applied to subduction zones and strike-slip configurations, though computational costs remain prohibitive due to the absence of efficient algebraic solvers.

For variational approaches, Pipping (2014) and Pipping et al. (2015) established existence and uniqueness results for unilateral frictional contact with Dieterich's law using fixed-point arguments. The extension to multibody layered systems, however, introduces essential geometric nonlinearities arising from large relative tangential displacements that have not been systematically addressed in the variational context.

The present work makes the following principal contributions:

- **Generalized variational framework.** We extend the variational approach of Pipping et al. (2016) to layered fault systems with $I \geq 2$ viscoelastic bodies, deriving a mathematical model that accommodates large tangential displacements through contact mapping techniques while preserving the convex structure essential for numerical analysis.
- **Decoupled solution strategy.** We prove that the coupled system of non-smooth convex minimization problems arising from Newmark time discretization admits decoupling via fixed-point iteration into independent rate and state subproblems, each possessing unique solutions under standard assumptions.
- **Convergence of TNNMG methods.** For the spatially discretized rate problems with given state, we establish global convergence of truncated non-smooth Newton multigrid methods, extending the abstract theory of Gräser and Sander (2019) to the mortar-discretized multibody setting.
- **Robust numerical verification.** Numerical experiments on spring-slider ($I = 2$) and layered ($I = 5$) configurations demonstrate that solver performance—measured in fixed-point iterations and multigrid cycles—remains essentially independent of the number of faults, confirming the theoretical predictions.

The remainder of this paper is organized as follows. Section 2 introduces the mathematical model for layered fault systems with RSF, including weak formulation and well-posedness. Section 3 presents semi-discretization in time and proves existence and uniqueness for the decoupled subproblems. Section 4 addresses full discretization in space and time, with emphasis on the mortar method and piecewise-constant state approximation. Section 5 analyzes the algebraic solution via TNNMG methods and establishes convergence. Section 6 reports numerical experiments, and Section 7 presents conclusions and directions for future work.

METHOD

Mathematical Model

Layered Fault System with Rate- and State-Dependent Friction: Let $\Omega \subset \mathbb{R}d$ ($d = 2, 3$) denote the reference domain of a geological structure decomposed into I non-overlapping Lipschitz subdomains Ωi, $i = 1, \ldots, I$, satisfying $\Omega = \cup_{i=1}^{T} \Omega i$. The subdomains are layered such that adjacent pairs share fault interfaces

$$\Gamma_{i,i+1}^{f} = \Omega_i \cap \Omega_{i+1}, \quad i = 1, \ldots, I-1,$$

with all other intersections being empty. The boundary of each subdomain decomposes disjointly into Dirichlet, Neumann, and contact parts, respectively. We denote the global contact interface by $\Gamma F = \cup_{i=1}^{T-1} \Gamma_{i,i+1}^{f}$.

Assuming the absence of fault opening (Assumption 2.2), the actual contact surface evolves with the deformation. Large tangential displacements are accommodated through a contact mapping πu: $\Gamma^{Bf,u}_{j} \rightarrow \Gamma^{Tf,u}_{j}$ that parameterizes the top reference domain over the bottom. The jump of a piecewise-defined field across the deformed contact boundary is defined by

$$[\![\not{f}]\!]^{u} = \not{f}^{B} - \not{f}^{T} \circ \pi^{u} \text{ on } \Gamma^{Bf,u}_{j}.$$

Rate- and State-Dependent Friction Law: The friction law takes the subdifferential form: $-\sigma t \in \partial[\![^{uuu}$ $\phi\, ([\![^{u}]\!]^{u}, \alpha)$, where σt is the tangential traction and $\phi\, (\cdot, \alpha)$ is a state-dependent convex functional. The state α evolves according to $-\dot{\alpha} \in \partial\alpha\psi\, (\alpha, |[\![^{u}]\!]^{u}|)$, with $\psi\, (\cdot, |[\![^{u}]\!]^{u}|)$ convex.

The classical RSF coefficient (Ruina, 1983) is given by

$$\mu\, (V, \theta) = \mu_0 + a\cdot ln(V/V_0) + b\cdot ln(V_0\theta/L),$$

where μ_0, V_0, a, b, $L > 0$ are material parameters. State evolution follows either Dieterich's aging law, $\dot{\theta} = 1 - V\theta/L$, or Ruina's slip law, $\dot{\theta} = -(V\theta/L) \cdot \ln(V\theta/L)$. Following Pipping et al. (2015), these translate into the variational framework via the transformed state $\alpha = \ln\theta$.

Strong and Weak Formulations: Problem 1 (Strong formulation). The governing equations for the layered fault system with RSF consist of the Kelvin–Voigt constitutive law $\sigma(u) = A\varepsilon(\dot{u}) + B\varepsilon(u)$, the balance of linear momentum $\rho\ddot{u} = \operatorname{div}\sigma(u) + f$ in $\Omega \setminus \Gamma F$, Dirichlet and Neumann boundary conditions on ΓD and ΓN respectively, frictional contact conditions on ΓF, and the state evolution equation on $\Gamma^{Bf,u}_{j}$. Initial conditions $u\,(0) = u_0$, $\underset{\sim}{u}\,(0) = \underset{\sim}{u}_0$, $\alpha\,(0) = \alpha_0$ are prescribed.

The weak formulation is stated on the Hilbert space $H_0 = \{v \in H^1(\Omega_1)^{d} \times \cdots \times H^1(\Omega^{T})^{d} \mid v = 0 \text{ on } \Gamma^{K}\}$ with the affine subspace H_0^{u} enforcing weak non-penetration. The variational inequality for the velocity reads: find $\underset{\sim}{u} \in H_0^{u}$ such that

$$\langle \rho\ddot{u}, v - \underset{\sim}{u} \rangle + a\, (\underset{\sim}{u}, v - \underset{\sim}{u}) + b\, (u, v - \underset{\sim}{u}) + \Phi^{u}\, (v, \alpha) - \Phi^{u}\, (\underset{\sim}{u}, \alpha) \geq l\, (v - \underset{\sim}{u}) \;\; \forall v \in H_0^{u},$$

coupled with the state inequality: find $\alpha \in L^2(\Gamma^{f})$ such that

$$\langle \dot{\alpha}, \beta - \alpha \rangle^{L^2(\Gamma^f)} + \Psi^{u}\, (\beta, \underset{\sim}{u}) - \Psi^{u}\, (\alpha, \underset{\sim}{u}) \geq 0 \quad \forall \beta \in L^2(\Gamma^f),$$

for a.e. $t \in (0, T_0)$. Here, $a\,(\cdot, \cdot)$ and $b\,(\cdot, \cdot)$ denote the viscous and elastic bilinear forms, $l\,(\cdot)$ the load functional, and Φ^{u}, Ψ^{u} the convex integral functionals induced by ϕ and ψ, respectively.

Semi-Discretization in Time

Temporal Discretization via the Newmark Scheme: We partition $[0, T_0]$ into N time steps $0 = t_0 < \cdots < t^{!} = T_0$ with step sizes $\tau n = t^{n}_{+1} - t^{n} > 0$. Applying the classical Newmark scheme (Hairer, Lubich, & Wanner, 2006) and freezing the geometric nonlinearity at the previous step, Problem 1 reduces at each time step to the following pair of decoupled convex minimization problems.

Problem 3 (Time-discrete rate problem). Find $\underset{\sim}{u}^n \in H_0^{un-1}$ minimizing

$$\mathcal{V}(v, \alpha) = \tfrac{1}{2}\, a^n (v, v) + \Phi^{un-1} (v, \alpha) - l^n(v),$$

where $a^n (\cdot, \cdot)$ is the Newmark-modified bilinear form and $l^n (\cdot)$ the effective load functional at step n.

Problem 4 (Time-discrete state problem). Find $\alpha^n \in L^2(\Gamma^f)$ minimizing

$$\varepsilon\, (\beta, \underset{\sim}{u}) = \tfrac{1}{2}\, \|\beta\|^{2L^2(\Gamma^f)} + \tau\, \Psi^{Bn-1} (\beta, \underset{\sim}{u}) - \langle \alpha^{n-1}, \beta \rangle^{L^2(\Gamma^f)}.$$

Proposition 3.3 (Existence and uniqueness—rate problem). Under Assumptions 2.1–2.4 and given $\alpha \in L^2(\Gamma^f)$, Problem 3 has a unique solution $\underset{\sim}{u}^n$ for each time step n. The proof employs Korn's second inequality, lower semi-continuity of Φ^{un-1} (Fonseca & Leoni, 2007, Theorem 6.49), and the Lax–Milgram theorem (Glowinski, 1984, Lemma 4.1).

Proposition 3.4 (Existence and uniqueness—state problem). For both Dieterich's and Ruina's laws, Problem 4 admits a unique solution $\alpha^n \in L^2(\Gamma^f)$ for any given $\underset{\sim}{u} \in H_0^{un-1}$ (Pipping et al., 2015, Proposition 4.4; Glowinski, 1984, Lemma 4.1).

Full Discretization in Space and Time

Finite Element Spaces and Mortar Discretization: For each subdomain Ωi, let ${}^{3}{}_{i}$ be a shape-regular simplicial triangulation. The conforming piecewise-linear finite element space is $Si = \{v \in C(\Omega_i)^{d} \mid v|T \in P_1(T)^d, v|\Gamma^f{}_i = 0\}$, with product space $S = S_1 \times \cdots \times S^T \subset H_0$. Triangulations need not match at fault interfaces.

The dual mortar method (Wohlmuth, 2000) introduces dual basis functions φq, $q \in$ ᴘ^f, satisfying the bi-orthogonality condition $\langle \lambda_p|\Gamma^f, \varphi$□$\rangle = \delta_p$□. This induces the direct splitting S = я ⊕ z into a "conforming" part я (weakly satisfying non-penetration) and a "slip" part z, which further decomposes as $z = z^n \oplus zt$ according to approximate normal and tangential directions. The discrete solution space is S_0^{un-1} = я ⊕ zt.

For the state variable, we employ piecewise-constant functions $\mathscr{B}^f$ on the dual partition $\{Cp \mid p \in$ ᴘ$^f\}$ of the fault triangulation. The nodal approximation of the friction functional is $\Phi^B (v, \alpha) = \Sigma p\, \phi_p([\![v]\!]_p)$, using the nodal jump value $[\![v]\!]_p$ and node-averaged state α_p.

Algebraic Solution and Convergence Analysis

Truncated Non-smooth Newton Multigrid (TNNMG): The discrete rate problem minimizes the functional $\mathcal{V}^B(v) = \tfrac{1}{2}\, a^n (v, v) - l^{n}_{j}{}^{B}(v) + \Sigma_p\, \phi_p(v_p)$ over S_0^{un-1}. The nonsmooth separable structure over the nodal coefficients $v_p \in \mathbb{R}^d$ allows the application of the TNNMG algorithm of Gräser and Sander (2019).

Algorithm 1 (TNNMG iteration). Given iterate $u^v \in S_0^{un-1}$:

- **Presmoothing:** $\bar{u}^v = P(u^v)$ via inexact nonlinear Gauss–Seidel sweeps on the local subspaces яp.
- **Coarse correction:** Compute the Newton correction δu^v by solving the restricted quadratic problem on the truncated subspace $z(\bar{u}^v)$ = я + span $\{\lambda_p\, x \mid x \in T_p{}^B, [\![u^v]\!]_p \neq 0\}$.

- **Monotone line search:** Set $u^{\nu}_{+1} = \bar{u}^{\nu} + \rho\,(\bar{u}^{\nu}, \delta u^{\nu})\,\delta u^{\nu}$ with step size ρ ensuring $\mathcal{V}^{B}(u^{\nu}_{+1}) \leq \mathcal{V}^{B}(\bar{u}^{\nu})$.

Theorem 5.2 (Global convergence of TNNMG). For any initial iterate $u_0 \in S_0^{un-1}$, the sequence $\{u^{\nu}\}$ generated by Algorithm 1 converges to the unique solution of the discrete rate problem.

Proof. The functional $\mathcal{V}^{B}$ is strictly convex and coercive on S_0^{un-1}. The presmoothing step provides energy descent by construction. On the truncated subspace $z(\bar{u}^{\nu})$, the restriction of $\mathcal{V}^{B}$ is twice continuously differentiable, yielding a well-defined Newton direction. Global convergence follows from Gräser and Sander (2019, Corollary 4.5) combined with Theorem 5.6 and Lemma 5.8 of the same reference for inexact presmoothing. ■

RESULTS AND DISCUSSION

General Experimental Setup: All experiments are performed in $d = 2$ dimensions. Material and friction parameters are summarized in Table 1. The loading velocity at the upper boundary is $v^{K}\xi(t)\cdot e_1$ with $v^{K} = 2 \times 10^{-4}$ m/s and a smooth transition function $\xi(t)$.

Table 1. Material and Friction Parameters

Bulk Parameter	Value	Friction Parameter	Value
Bulk modulus E	4.12×10^{7} Pa	Reference velocity V_0	1×10^{-6} m/s
Poisson's ratio ν	0.3	Reference friction μ_0	0.6
Density ρ	5×10^{3} kg/m³	Direct effect a	0.010
Gravitational acceleration g	9.81 N/kg	Evolutionary effect b	0.015
—	—	Characteristic slip distance L	0.02 m

Source: Parameters adopted from Ampuero & Rubin (2008) and Ruina (1983).

Adaptive Time-Stepping Strategy: For the solution at t^{n}, an initial step size $\tau^{n*} = \tau^{n}_{-1}$ is proposed. Two candidate solutions are computed: one by a single step of size $2\tau^{n*}$ and one by two steps of size τ^{n*}. If the $L^{2}(\Gamma^{f})$-discrepancy in the state variable exceeds the threshold $\delta\tau = 10^{-5}$, the step size is halved; otherwise, it is doubled up to the maximal step size. This error-based control efficiently resolves the multiple-order-of-magnitude velocity variation during slip events without excessive refinement during interseismic phases.

Spring-Slider Configuration (I = 2): The spring-slider configuration consists of two viscoelastic bodies of dimensions 5 m × 1 m sharing the fault interface $\Gamma^{f} = (-2.5, 2.5) \times \{0\}$. After $K = 5$ adaptive refinements, the final triangulation comprises 1,274 vertices with mesh diameters $4.4 \text{ cm} \leq h^{J} \leq 70.8$ cm.

Result 6.1 (Periodicity of slip events). Following the initial loading phase (approximately 20 s), 26 nearly periodic slip events are observed, with slip velocities spanning ten orders of magnitude from 10^{-11} to 10^{-3} m/s. Events exhibit bilateral nucleation from the fault center, with identifiable foreshock and aftershock sequences consistent with physical observations.

Result 6.2 (Solver performance). Adaptive time-stepping reduces τ^n by approximately two orders of magnitude during slip events. Fixed-point iterations remain at 2–4 per time step throughout the simulation. Total multigrid cycles per time step range from 5 to 29, with no systematic dependence on event magnitude, confirming the robustness of the algebraic solver.

Layered Fault System (I = 5): The five-body configuration comprises bodies of thicknesses 1, 0.3, 0.09, 0.3, and 1 m over a horizontal extent of 5 m, with four fault interfaces. The final triangulation after K = 5 refinements contains 4,057 vertices with mesh diameters 3.2 cm $\leq h^J \leq$ 70.8 cm.

Result 6.3 (Multiscale behavior). The uppermost fault, Γ_{45}^f, exhibits periodic slip events qualitatively analogous to the spring-slider, validating model consistency. The intermediate faults Γ_{23}^f and Γ_{34}^f display oscillatory loading with amplitudes several orders of magnitude smaller—a behavior requiring further physical and numerical investigation. The lowest fault, Γ_{12}^f, exhibits stable creep punctuated by velocity jumps coinciding with upper-fault events.

Result 6.4 (Solver scalability). The number of fixed-point iterations per time step (2–4) and the upper bound on multigrid cycles (25) remain virtually identical to those of the spring-slider experiment. This confirms that the computational complexity of the proposed solver scales independently of the number of fault interfaces, in agreement with the theoretical convergence analysis.

CONCLUSION

This study has presented a comprehensive variational framework for multiscale geological fault systems with rate- and state-dependent friction, extending the subduction zone model of Pipping et al. (2016) to layered multibody configurations with an arbitrary number of non-intersecting fault interfaces. The combination of energy-conserving Newmark time discretization, dual-mortar finite element methods, and truncated non-smooth Newton multigrid solvers yields a numerically robust approach whose per-step computational complexity remains bounded independently of the number of faults. The principal theoretical contributions of this work are: (i) the derivation of a decoupled convex structure that preserves existence and uniqueness for both the rate and state subproblems; and (ii) the proof of global convergence for TNNMG methods in the mortar-discretized multibody setting, generalizing the abstract theory of Gräser and Sander (2019). The numerical experiments demonstrate that adaptive time-stepping efficiently captures slip events across velocity ranges exceeding ten orders of magnitude, while algebraic solver performance remains stable as the number of fault interfaces increases. Future research directions include: (i) extension of the framework to three spatial dimensions with parallel multigrid implementation; (ii) rigorous existence theory for the fully coupled continuous problem with Ruina's slip law; (iii) physical and numerical investigation of the oscillatory behavior observed on intermediate fault interfaces; and (iv) incorporation of fault opening through unilateral non-penetration constraints and dynamic activation and deactivation of friction laws.

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